REMARKS

At page 2 of the present Office Action of May 4, 2004, the Examiner requested the re-submission of three articles referred to by the Applicants and pages 1-4 of the Amendment filed on March 4, 2003.

Applicants submit that the following three articles accompany the present reply.

1. Aki-Hiro Sato, and Hideki Takayasu, "Dynamic numerical models of stock market price: from microscopic determinism to macroscopic randomness" Physica A, 250(1998), 231-252.

2. Hideki Takayasu, Aki-Hiro Sato, and Misako Takayasu, "Stable infinite variance fluctuations in randomly amplified Langevin systems" Phys. Rev. Lett,, 79(1997), 9676-969.

3. H. Takayasu and M.. Takaysu, "Critical fluctuations of demand and supply", Physica A 269 (1999) 24-29.

Additionally, Applicants have attached a copy of the previous Amendment filed on March 4, 2004. This Amendment includes claims 25-30.

Applicants wish to thank the Examiner for withdrawing the rejection of claims 25-30.

Applicants wish to thank the Examiner for stating that in response to his request for such information, this file will be given priority status.

CONCLUSION

Applicants have made a diligent effort to provide information as required by the Examiner. Early and favorable reconsideration of this information and the claimed invention are respectfully requested.

Respectfully submitted,

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Physica A 250 (1998) 231–252

Dynamic numerical models of stock market price: from microscopic determinism to macroscopic randomness

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Dynamic numerical models of stock market price: from microscopic determinism to macroscopic randomness

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Abstract

A variant of threshold dynamics is introduced to model the behaviors of a large assembly of dealers in a stock market. Although the microscopic evolution dynamics is deterministic the collective behaviors such as market prices show seemingly stochastic fluctuations. The statistical properties of market price change can be well approximated by a simple discrete Langevin-type equation with random amplification. The macroscopic stochastic equation is solved both numerically and analytically showing that the market price change generally follow power-law distributions in the steady state. The reason for the appearance of rapid decay in the distribution tails are discussed. © 1998 Published by Elsevier Science B.V. All rights reserved.

PACS: 02.50.-r; 05.45.+b; 05.40.+j
Keywords: Stock market; Threshold dynamics; Langevin-type equation; Power-law distribution

1. Introduction

Human activity is generally based on one's intention and is hardly described by mathematical laws. However, a large assembly of human activities sometimes cancel individuality and produce macroscopic behaviors which can be analyzed by statistical physics approaches to some extent.

A good example can be found in the Internet activity. Millions of users are independently sending or receiving information packets from peripheral terminals of the computer network. Although every packet has its own role reflecting the user's intention, the statistics of packet density at higher level routers in the Internet clearly

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shows a universal scaling property such as fractality [1], which implies the existence of underlying mechanism of dynamic phase transition due to information congestion [2].

Recently, topics categorized in economics are attracting much attention of statistical physicists, such as company growth rates [3], distribution of company sizes [4], and fluctuation of stock market prices [5]. These quantities are not called "macro-variables" in standard economics, but they can be regarded as "macro-variables" in statistical physics sense. For example, the power spectrum of any stock market prices universally follows an inverse square power law with respect to the frequency, showing that each price change can be regarded as an independent stochastic event thus price predictions are gambles.

Several years ago one of the authors (H.T.) and coworkers introduced a simplified numerical model of stock market prices and clarified a possible origin for the independent fluctuations [6]. In the model a deterministic dynamics is assumed for an assembly of agents describing mutual trades by a threshold dynamics including nonlinear irreversible interactions. It is shown that nonuniformity among agents' behaviors and the nonlinear interaction create chaotic fluctuations having inverse square power spectrum. The maximum Lyapunov exponent is estimated to be zero indicating that the system is at the edge of chaos [7].

In view of statistical physics an open problem in market-price changes is the reason for the large deviation from Gaussian distribution. As firstly pointed out by Mandelbrot [8] and recently confirmed with much higher accuracy by Montegna and Stanley [5] that the distribution of price changes are well approximated by a Lévy's stable distribution with characteristic exponent about 1.4. The observed distribution function has power tails consistent with the stable distribution for a wide range, however, accompanied with exponential decays for very large absolute values.

In this paper we focus our attention on the distribution of stock market price changes. A modified numerical model of interacting agents based on deterministic threshold dynamics is introduced as the starting point of our theoretical approach in the following section. By numerical simulation we show that the deterministic system produces seemingly stochastic market price changes which show consistent properties with known empirical features. In Section 3 we derive a simple stochastic equation for the macro-variable, the market price change. We clarify basic properties of this stochastic equation in Section 4. We prove that the market-price changes in the model rigorously follow a power-law distribution. In Section 5 we discuss the origin of quick decay for large fluctuations. We propose two possibilities: nonlinearity in trades and finiteness of data numbers. The final section is devoted to the concluding remarks.

2. Formulation of the threshold model

The complex fluctuations of market prices have been attributed to various factors such as fundamentals or political decisions, however, in this paper we pay attention only to the process of selling or buying stocks in a market, which is one of the simplest interactions between dealers, in order to explain the complex behavior as simple as possible. A threshold model of stock market exchange based on dealers' dynamics was recently developed by Hirabayashi et al. [6]. Here we introduce a revised model of stock market fluctuation based on the previous model.

In general, a dealer determines to sell (or buy) stocks if a market price is higher (or less) than a price for selling (or buying) in his mind. In real stock markets, each dealer determines prices to sell or to buy for each brand. However, we consider a stock market dealing with only one brand for simplicity. We denote the ith dealer's threshold prices to sell and to buy stocks by S_i and B_i, respectively. The difference between S_i and B_i, $L_i \equiv S_i - B_i$ must be positive. The ith dealer can buy stocks from the jth dealer if $B_i \geqslant S_j$ is satisfied. Also for simplicity we fix L_i to a constant value Λ independent of the dealer number $\{i\}$. Namely, in the present system a trade can be taken place when the following condition holds:

$$\max\{B_i\} - \min\{B_i\} \geqslant \Lambda \,, \tag{1}$$

where $\max\{\dots\}$ and $\min\{\dots\}$ denote the maximum and minimum value in the set of dealers' buying threshold $\{B_i\}$. We assume a trade occurs between the two dealers who propose the maximum buying price and the minimum selling price.

The market price, $P(t)$, is defined by the mean value of $\max\{B_i\}$ and $\min\{S_i\}$ when a trade occurs. If condition (1) is not satisfied, there occurs no trade and the market price does not change.

$$P(t) = \begin{cases} (\max\{B_i\} + \min\{S_i\})/2 & \text{condition (1) is satisfied,} \\ P(t-1) & \text{otherwise.} \end{cases} \tag{2}$$

In the threshold model each dealer changes his price in a unit time by the following deterministic rule:

$$B_i(t+1) = B_i(t) + a_i(t) + c_i\{P(t) - P(t_{prev})\} \,, \tag{3}$$

where $a_i(t)$ denotes the ith dealer's expectation of bid price on the time t, t_{prev} denotes the time when the last trade occurred and c_i is a coefficient, which shows the ith dealer's response to the market-price change.

Initial conditions of $B_i(t)$ and $a_i(t)$ are given by uniform random numbers in the range $(-\Lambda, \Lambda)$ and $(-\alpha, \alpha)$, respectively, where α is a given parameter. For simplicity, we consider c_i to be a constant $c \geqslant 0$ independent of i. As known from Eq. (3) we assume that all the dealers' threshold prices slide by the same amount, $c\{P(t) - P(t_{prev})\}$, until the next trade occurs. The system size, which is the number of dealers, is denoted by N. As Eq. (3) is linear and translationally invariant the absolute values of $\{B_i\}$ are not meaningful, and market-price fluctuations of our model is independent of the place of the origin of $\{B_i\}$.

The dynamics of $a_i(t)$ characterizes the behavior of the ith dealer: when $a_i(t)$ is positive (or negative) the ith dealer increases (or decreases) his price in mind meaning that he wants to sell (or buy) stocks. In the previous model [6] we assigned a limit

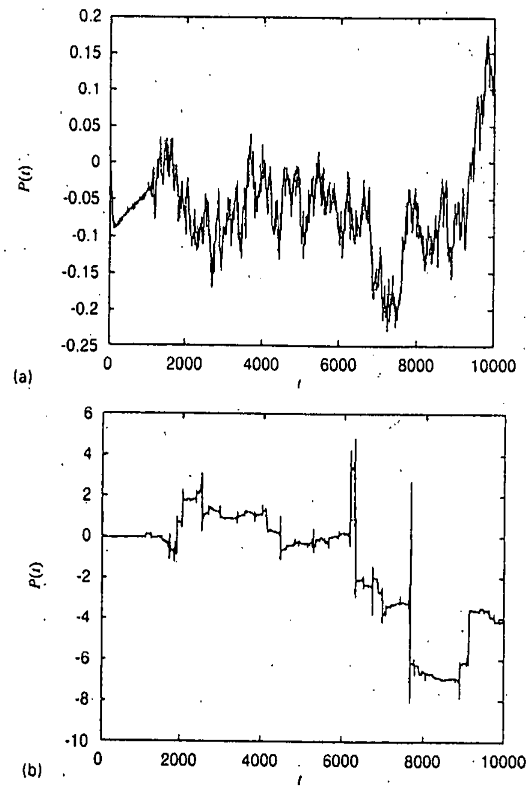

Fig. 1. Examples of temporal fluctuations of market price, $P(t)$: (a) $c=0$; (b) $c=0.3$.

that all dealers have infinitely much amount of property, and the sign of $a_i(t)$ does not change meaning that a buyer (or seller) is always a buyer (or seller). Here, we assume the opposite limit situation that all dealers have small amout of property and a dealer changes his position from a buyer (or seller) to a seller (or buyer) after he buys (or sells) his stocks, namely, the sign of $a_i(t)$ changes whenever the ith dealer is involved in a trade. The absolute value of $a_i(t)$ represents his temper and it does not change as represented in the following dynamics:

$$a_i(t)=\begin{cases} -a_i(t-1) & \begin{cases}\text{when } a_i(t-1)>0 \text{ for the buyer,}\\ \text{or}\\ \text{when } a_i(t-1)<0 \text{ for the seller,}\end{cases} \\ a_i(t-1) & \text{others.}\end{cases} \tag{4}$$

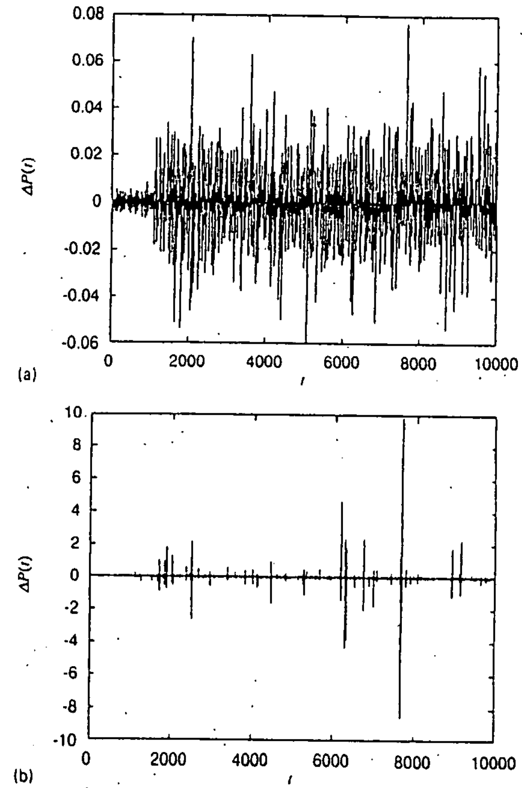

Fig. 2. Examples of temporal fluctuations of price changes, $\Delta P(t)$: (a) $c=0$; (b) $c=0.3$.

This dynamics has been introduced by Hirabayashi [9] and its chaotic time evolution is studied for the case of small N.

We will discuss general relations between system features and parameters in the following sections. Before discussing details we roughly examine system features by computer simulations. Throughout our numerical simulations we fix the parameters $N=100$, $\alpha=0.01$ and $\Lambda=1.0$ and regard c as the only control parameter.

We show two examples of temporal market price evolution for $c=0.0$ and a larger c, $c=0.3$ in Fig. 1, respectively. We find the fluctuations reach statistically steady states after transitory initial responses nearly independent of the value of c. It is found that the length of transitory response depends on the initial configuration. As far as we checked carefully the length of transitory responses were always smaller than 5000 steps. In the following numerical analyses we ignore the first 5000 steps whenever we observe steady-state properties.

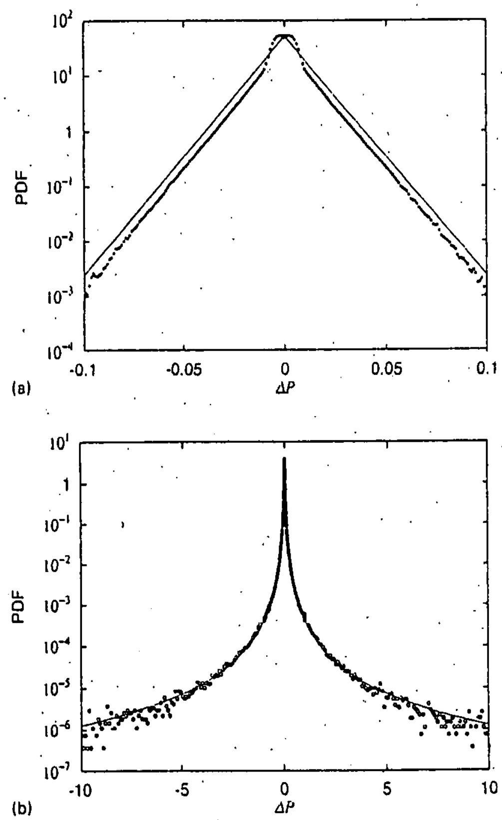

Fig. 3. Semi-log plot of the PDFs of $\Delta P(t)$: (a) $c=0$; (b) $c=0.3$. Dots represent the numerical simulation values and lines represent theoretical curves; (a) a hybrid of Gaussian–Laplacian distribution, whose variance is 0.001; (b) a power-law distribution, $f(\Delta P) \propto (\Delta P)^{-2.5}$.

Following the way of real market analysis by Montegna and Stanley [5], we observe market-price changes,

$$\Delta P(t) = P(t) - P(t-1). \tag{5}$$

and estimate the probability density function (in short PDF) of $\Delta P(t)$. In Fig. 2 we plot the price changes for the cases of $c=0.0$ and $c=0.3$. For calculation of PDFs we observe price changes for time steps up to 5×10^6. We show PDFs for $c=0.0$ and $c=0.3$, respectively, in Figs. 3a and 3b. In the case of smaller c (from 0 to about 0.1) the PDF can be approximated by a hybrid distribution of a Gaussian distribution (for small $|\Delta P|$) and of a Laplacian distribution (for larger $|\Delta P|$). For larger c (from about 0.1 up to about 0.45) the PDF is approximated by a power law. For larger values

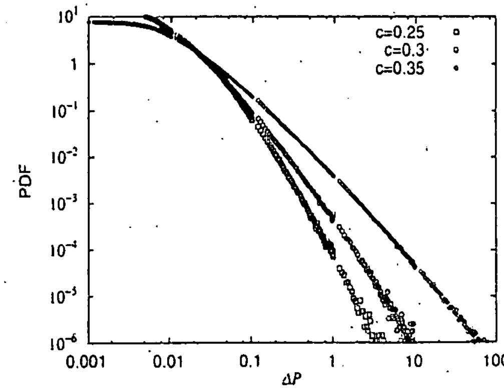

Fig. 4. Log–log plot of PDFs of $\Delta P(t)$ for $c=0.27, 0.31$ and 0.35.

of c the distributions have longer tails and the exponents of power-law distribution are estimated to be smaller as shown in Fig. 4. For c larger than 0.45 price fluctuations are very unstable and diverge quickly, namely we cannot observe any steady distribution.

The PDF looks similar to the distribution of price changes for real stock market reported by Montegna and Stanley [5] in the case of c around 0.3 except the tail parts for very large $|\Delta P|$. We will discuss the reason why the distribution in real data has rapid decays on larger scale in Section 5.

We are going to show that this threshold model is approximated by a simple stochastic process, which is the discrete version of Langevin equation in the next section. By the discrete Langevin equation we clarify the mechanism of generating power-law fluctuations.

3. Macroscopic stochastic process

As clarified in the preceding section, the parameter c in Eq. (3) mainly controls the fluctuations. In order to clarify the effect of c more clearly we introduce a simplified stochastic dynamics for changes of market prices.

We denote $\tau(s)$ as the time when the sth trade occurs. A conceptual illustration is shown in Fig. 5. There is no trade from $\tau(s)$ to $\tau(s+1)-1$, and market prices do not change, so that $P(\tau(s)) = P(\tau(s)+1) = \cdots = P(\tau(s+1)-1) \neq P(\tau(s+1))$. $\tau(s+1)$ denotes the time when the $(s+1)$th trade occurs. By definition $t_{prev} = \tau(s-1)$, and we have $P(t) - P(t_{prev}) = P(\tau(s)) - P(\tau(s-1))$. Here, we consider price changes at the sth trade as

$$\Delta p_s \equiv P(\tau(s)) - P(\tau(s-1)). \tag{6}$$

Since $c\Delta p_s$ is repeatedly added to the market price from time step $\tau(s)$ to $\tau(s+1)-1$, $cn_s\Delta p_s$ is added to the next change of price, Δp_{s+1}, where $n_s \equiv \tau(s+1) - \tau(s)$

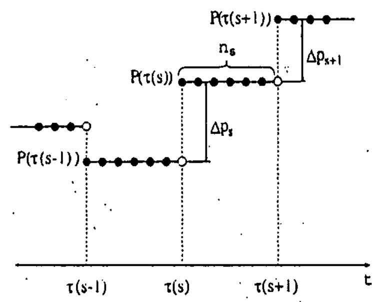

Fig. 5. A conceptual illustration of microscopic market-price fluctuations.

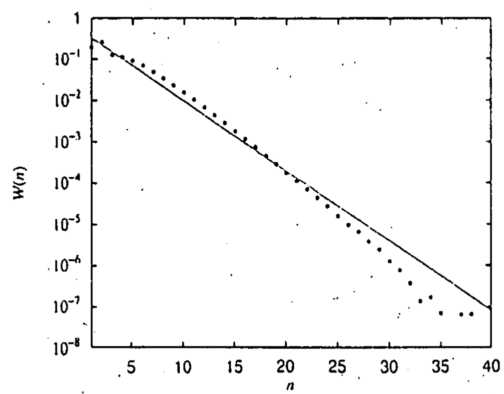

Fig. 6. Semi-log plot of the probability density of the interval between two successive trades, n_s. Dots represent the numerical simulation values and the line represents the exponential distribution fitted numerically.

gives the interval between trades. The distribution of n_s is independent of c and directly observed in our simulation as is shown in Fig. 6. The distribution can be well approximated by a discrete exponential distribution in the range $[1,\infty)$

$$W(n)=\sum_{k=1}^{\infty}\frac{1-e^{-\gamma}}{e^{-\gamma}}\exp(-\gamma k)\delta(n-k)\,, \tag{7}$$

where $\gamma=0.389$ is estimated from the slope.

There exists a spontaneous price fluctuation at each trade, which can be observed even when $c=0$. We define the spontaneous fluctuation with $c=0$ at the sth trade as ϕ_s and denote its distribution as $U(\phi)$. The distribution of ϕ_s plotted in Fig. 3a can be well approximated by a Laplace distribution as mentioned in the preceding section,

$$U(\phi)=\frac{1}{2\sigma}\exp(-|\phi|/\sigma)\,, \tag{8}$$

where σ is a scaling constant, $\sigma=0.01$. By assuming that Δp_s, n_s and ϕ_s are independent of each other we can approximate the price change by the following simple stochastic process:

$$\Delta p_{s+1}=cn_s\Delta p_s+\phi_s\,. \tag{9}$$

This is a kind of discrete Langevien equation. Such a linear stochastic equation containing additive and multiplicative noise has been derived in various fields; bending modes of the polymer with additive and multiplicative randomly external forces [10], motion of a passive scalar in turbulent flow [11,12], noise in dye lasers [13,14] and general large assemblies of nonlinear dynamical elements [15,16]. There are also intensive analyses on this type of equation in the mathematical study of on–off intermittency [17,18] and random multiplicative processes [19,20]. It has been expected that one-component stochastic equation with both additive and multiplicative noise has two types of behaviors [21]. In one regime the PDF has power-law tails, and in the other regime the PDF is characterized by a stretched exponential decay [22]. We analyze Eq. (9) in the framework of the characteristic function in the following sections.

4. Analysis of the stochastic process

As known from the above discussion our deterministic threshold model is divided into two parts: a microscopic generator of the spontaneous noise, and a macroscopic stochastic dynamics involving additive and multiplicative noises. In this section we discuss details of each part. As for the noise generator we clarify the dependence on the system parameters. For macroscopic dynamics we study steady-state statistics of Eq. (9).

4.1. *Statistics of the noise generator*

We have introduced two stochastic variables, n_s and ϕ_s in the previous section, in order to approximate features of our microscopic model described by Eq. (9). It is obvious that these stochastic variables are independent of c, so $\langle n\rangle$ and $\langle\phi^2\rangle$ are considered to be functions of parameters, N, α and Λ.

In the case $c=0$ the ith dealer changes his threshold price by a_i, per unit time and B_i moves nearly cyclically up and down between roughly the maximum and the minimum values of the set of $\{B_i\}$. The range of $\{B_i\}$ is expected to be about Λ, so we have the following estimation for the period T_i which characterizes the ith dealer's motion,

$$T_i\approx 2\Lambda/|a_i(t)|\,. \tag{10}$$

Taking an average over i, Eq. (10) gives the following relation:

$$\langle T\rangle\approx\frac{2\Lambda}{\langle|a|\rangle}\,. \tag{11}$$

The ensemble average of T_i, $\langle T \rangle$, and the system size N roughly determine the frequency of trades in the whole system, therefore, we may expect a relation

$$\langle n \rangle \propto \frac{\langle T \rangle}{N}. \tag{12}$$

In the case that the set $\{a_i\}$ is given by uniform random number in the range $(-\alpha, \alpha)$, we have $\langle |a| \rangle = \alpha/2$. Introducing Eq. (11) into Eq. (12) $\langle n \rangle$ is represented by Λ, N and α as

$$\langle n \rangle = m \frac{\Lambda}{N\alpha}, \tag{13}$$

where m denotes a proportional constant. By numerical simulations we estimate the proportional constant to be $m \approx 3.0$.

We can confirm the validity of Eq. (13) by observing the change of $\langle n \rangle$ as function of Λ or N or α in Fig. 7a–7c. Since n_s must be a positive integer by definition, $\langle n \rangle$ must be larger than unity. So the analytical estimation, Eq. (13), may be valid for $\langle n \rangle$ greater than unity. In fact, we can confirm this expectation from Fig. 7a, which shows a gap between numerical values and theory for $\langle n \rangle < 1$. However, for all other cases the numerical results fit very nicely to the roughly estimated relation, Eq. (13).

We can evaluate the variance of the spontaneous fluctuation ϕ_s, $\langle \phi^2 \rangle$ in the case of $c = 0$ in the following way. For $c = 0$ Eq. (3) becomes very simple as

$$B_i(t+1) = B_i(t) + a_i(t). \tag{14}$$

By definition the market price at the sth trade is given by

$$P(\tau(s)) = \{B_{i_s}(\tau(s)) + S_{j_s}(\tau(s))\}/2, \tag{15}$$

where $B_{i_s}(\tau(s))$ and $S_{j_s}(\tau(s))$ denote the buyer's and seller's prices, respectively. As schematically shown in Fig. 8, $B_{i_s}(\tau(s)) - B_{i_s}(\tau(s-1)) = |a_{i_s}| n_s$ for buyers, and $S_{j_s}(\tau(s)) - S_{j_{s-1}}(\tau(s)) = -|a_{j_s}| n_s$ for sellers. By the definition of trade, Eq. (1), we can expect that the buyer's price nearly equals to the seller's price at each trade. Noting that ϕ_s is given by the price change $P(\tau(s)) - P(\tau(s-1))$ we have a rough approximation for ϕ_s by assuming that the threshold prices whose dealers take a play in the next trade are similarly distributed independently:

$$\phi_s \approx (a_{i_s} - a_{j_s}) n_s / 2. \tag{16}$$

Calculating the second-order moments of both sides of Eq. (16) with the assumptions that $a_{i_{s+1}}$ and $a_{j_{s+1}}$ independently follow the uniform distribution in the range $[-\alpha, \alpha]$ and that n_s follows the discrete exponential distribution, Eq. (7), we obtain

$$\langle \phi^2 \rangle \propto 2m^2 \frac{\Lambda^2}{N^2} - m \frac{\Lambda\alpha}{N}. \tag{17}$$

We check this estimation of $\langle \phi^2 \rangle$ as functions Λ, N and α in Fig. 9. We find nice fits to Eq. (17) in all cases except Fig. 9b with large N and α. We estimate the

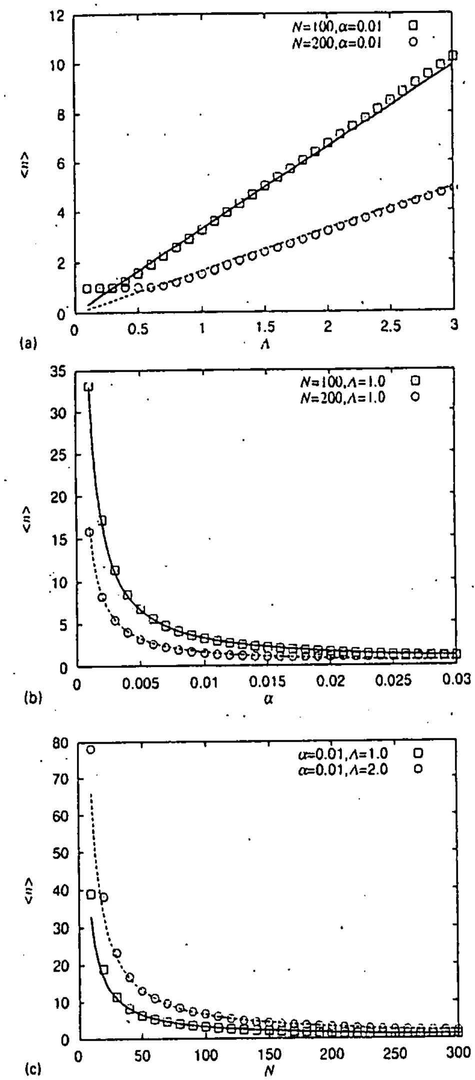

Fig. 7. The relations between $\langle n \rangle$ and the system parameters, Λ, α and N. Each figure is shown as function of one parameter with the other two parameters being fixed. Squares and cycles represent results of numerical simulations for each parameter set. Solid and dashed curves are given by analytical estimation, Eq. (13): (a) $\langle n \rangle$ versus α with $N = 100$, $\alpha = 0.01$ and $N = 200$, $\alpha = 0.01$; (b) $\langle n \rangle$ versus α with $N = 100$, $\Lambda = 1.0$ and $N = 200$, $\Lambda = 1.0$; (c) $\langle n \rangle$ versus N with $\alpha = 0.01$, $\Lambda = 1.0$ and $\alpha = 0.01$, $\Lambda = 2.0$.

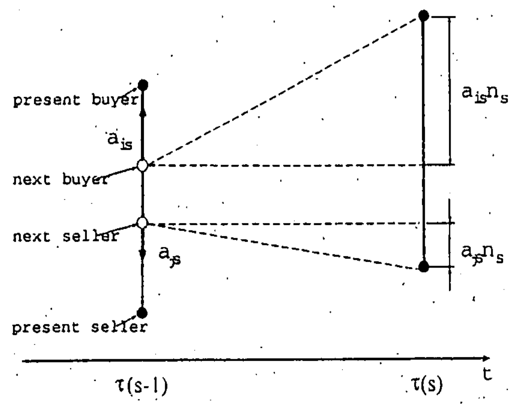

Fig. 8. A conceptual illustration representing trade interaction at time $\tau(s-1)$ and $\tau(s)$. Each dot represents a dealer's threshold buying/selling price.

proportional constant to be about 0.08 by fitting Eq. (17) with numerical results. We confirm this constant is invariant for different fixing parameter sets and different initial distributions. In the same way as the estimation of $\langle n \rangle$, Eq. (17) gives an adequate approximation for $\langle n \rangle \geqslant 1$. For large N and α as in the case of Fig. 9b $\langle n \rangle$ in Eq. (13) becomes smaller than unity, therefore, we have the deviation from the theory.

In Fig. 10 we check the validity of Eq. (16) which gives a correlation between ϕ_s and n_s. As estimated by Eq. (16), the range of ϕ is nearly proportional to n up to about $n = 15$. Therefore, Eq. (16) may give a nice approximation for smaller n.

4.2. The behavior of the macroscopic stochastic equation

We now consider the case of $c \neq 0$ and analyze the behavior of Eq. (9) theoretically [23]. We assume independence of the multiplicative and additive variables n_s and ϕ_s for simplicity.

We introduce the characteristic function, $Z(\rho,s)$, which is the Fourier transform of the probability density, $f(\Delta p,s)$:

$$Z(\rho,s) \equiv \langle e^{i\rho\Delta p} \rangle \equiv \int_{-\infty}^{\infty} e^{i\rho\Delta p} f(\Delta p,s) d(\Delta p)\,, \tag{18}$$

where $\langle \ldots \rangle$ denotes the average over all realizations. From Eq. (9) we obtain

$$Z(\rho,s+1) = \Phi(\rho) \int_{-\infty}^{\infty} W(n) Z(cn\rho,s) dn\,, \tag{19}$$

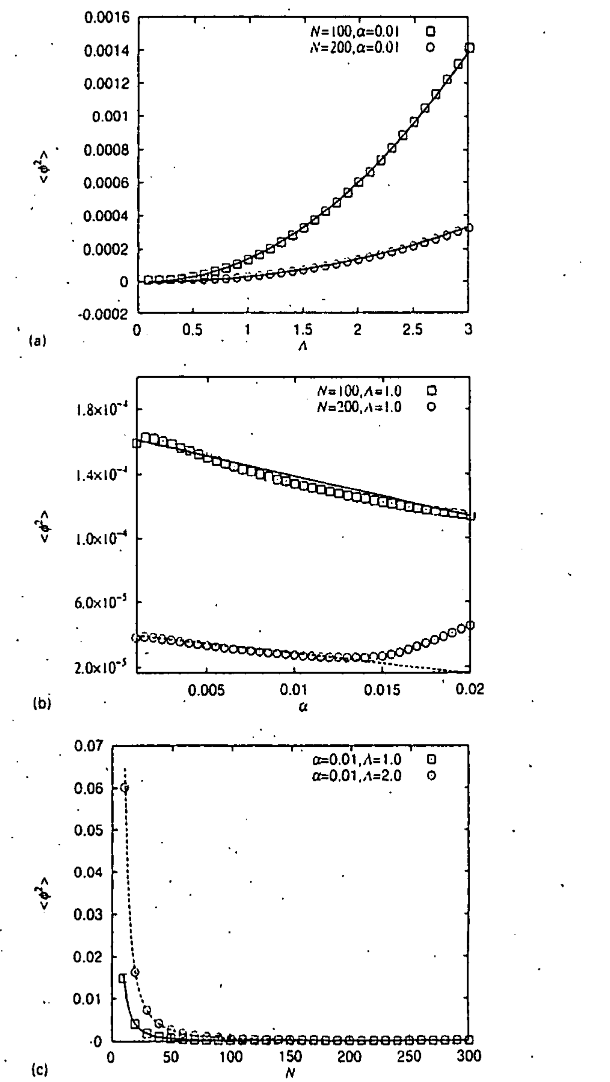

Fig. 9. The relations between $\langle \phi^2 \rangle$ and the system parameters, Λ, α and N. Each figure is shown as a function of one parameter with the others being fixed. Squares and circles represent results of numerical simulations. Solid and dashed curves are given by the analytical estimation, Eq. (17): (a) $\langle \phi^2 \rangle$ versus Λ; (b) $\langle \phi^2 \rangle$ versus α; (c) $\langle \phi^2 \rangle$ versus N.

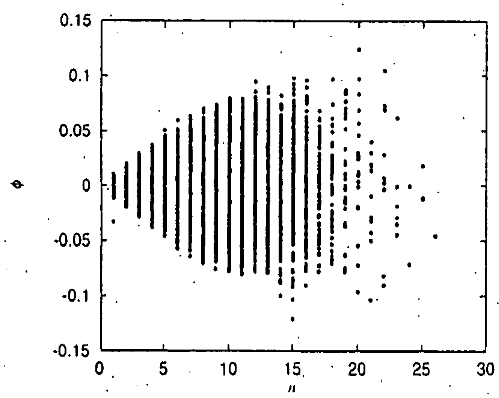

Fig. 10. Correlation between ϕ_s and n_s at $N=100$, $\alpha=0.01$ and $\Lambda=1.0$. Dots represent n_s and ϕ_s at the same trade.

where $\Phi(\rho) \equiv \langle e^{i\rho\phi_s} \rangle$ is the characteristic function for the additive variable, ϕ_s. As the distribution of ϕ is assumed to be stationary, its characteristic function, $\Phi(\rho)$ is represented by the set of moments, $\{\langle \phi^k \rangle, k=0,1,2,\ldots\}$ as $\Phi(\rho)=\sum_{k=0}^{\infty}\langle \phi^k \rangle \rho^k/k!$.

We can prove the uniqueness and stability of the steady solution of Eq. (19) in the following way [23]. Assuming the existence of steady solution, $Z^*(\rho)$, of Eq. (19) the deviation from the steady solution, $\bar{Z}(\rho)=Z(\rho,s)-Z^*(\rho)$, satisfies the same equation, Eq. (9), with a different boundary condition, $\bar{Z}(0,s)=0$. By taking absolute values of the equation we obtain an inequality:

$$|\bar{Z}(\rho,s+1)| \leqslant \max\{|\bar{Z}(\rho,s)|\}\,|\Phi(\rho)|\,, \tag{20}$$

where $\max\{\cdots\}$ represents the maximum value. Therefore, in the case $\Phi(\rho)<1$ for $\rho \neq 0$, which is satisfied whenever the additive noise is continuously distributed, the distribution of Δp converges to a unique steady solution if its exists starting from any initial distribution of $\{\Delta p_0\}$. As the equation for the steady solution, $Z^*(\rho)$, is given by

$$Z^*(\rho)=\Phi(\rho)\int_{-\infty}^{\infty} W(n)Z^*(cn\rho)dn\,, \tag{21}$$

we have the set of equations for moments, $\{\langle \Delta p^k \rangle, k=0,1,2,\ldots\}$ by assuming a Taylor expansion. Especially the second-order moment in the case of symmetric $U(\phi)$ satisfies the following relation:

$$\langle \Delta p^2 \rangle = c^2\langle n^2 \rangle \langle \Delta p^2 \rangle + \langle \phi^2 \rangle\,. \tag{22}$$

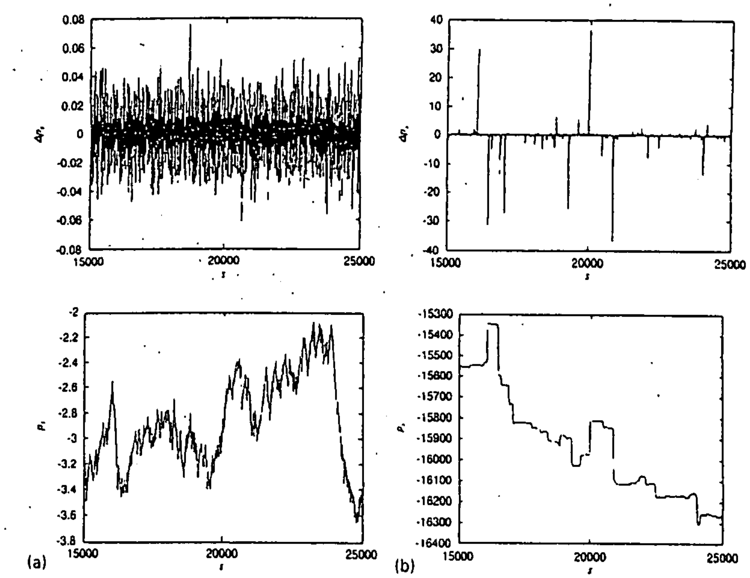

Fig. 11. Examples of temporal fluctuations for $c<c^*$ and $c\geqslant c^*$, where $c^*=0.25$: (a) Temporal fluctuations of price change Δp (top) and corresponding price P (bottom) for $c=0.1$. (b) Temporal fluctuations for $c=0.35$. Note that the scale on ordinate is much bigger than in (a).

For $c<c^*$, where $c^*=1/\sqrt{\langle n^2 \rangle}$, we have the stationary solution

$$\langle \Delta p^2 \rangle = \frac{\langle \phi^2 \rangle}{1-c^2\langle n^2 \rangle}\,. \tag{23}$$

On the other hand, for $c\geqslant c^*$ there is no stationary solution and it diverges as $s \to \infty$.

It is generally believed that no steady distribution exists when its second-order moment diverges, however, this is not logically correct. We can confirm the existence of steady distribution for $c\geqslant c^*$ by computer simulations as follows. We specify the distributions of n and ϕ for computer simulations and take c as the only control parameter. The parameters γ and σ in Eqs. (7), (8) are fixed as $\gamma=0.389$ and $\sigma=0.01$. The value of c^* is approximately given as $c^*=0.25$. In the numerical simulations the maximum time steps are typically 5×10^7 and we observe the distribution of $\{\Delta p\}$ for time steps after 1000. Typical examples of temporal fluctuations for $c<c^*$ and $c\geqslant c^*$ are shown in Fig. 11 (top). From definition of Δp_s, Eq. (6), market price can be calculated iteratively as shown in Fig. 11 (bottom).

For $c<c^*$ the fluctuations are obviously bounded, on the other hand for $c\geqslant c^*$, there sometimes appear very huge fluctuations. We can confirm the validity of Eq. (23) for the estimation of $\langle \Delta p^2 \rangle$ for $c<c^*$ as shown in Fig. 12. For $c>c^*$ the numerical

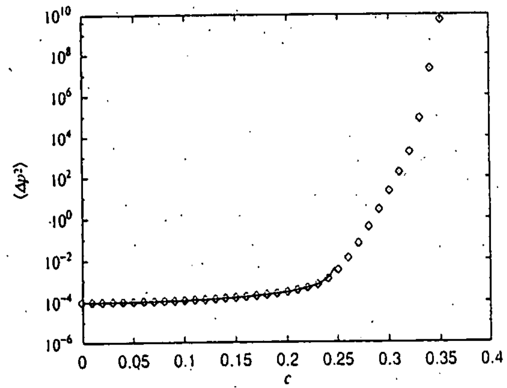

Fig. 12. Relation between $\langle \Delta p^2 \rangle$ and c at $t = 5 \times 10^7$. Dots represent numerical values obtained by the simulation and the curve shows the theoretical value given by Eq. (23).

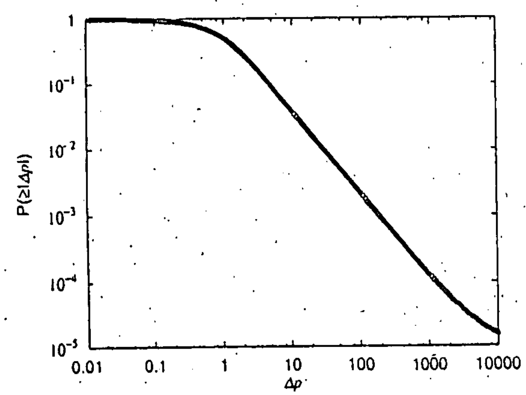

Fig. 13. Log–log plot of the cumulative distribution of Δp_t by the stochastic process Eq. (9) with $c = 0.3$. The distribution have a clear power-law tail.

value of $\langle \Delta p^2 \rangle$ tends to diverge but here we plot the value at the last time step, $t = 5 \times 10^7$.

We plot the cumulative distribution, which is defined as

$$F(\geq |x|) = \int_{-\infty}^{-|x|} p(x')\,dx' + \int_{|x|}^{\infty} p(x')\,dx' , \tag{24}$$

in Fig. 13, which clearly shows the existence of steady distribution with a power-law tail

$$F(\geq |\Delta p|) \propto |\Delta p|^{-\beta} , \tag{25}$$

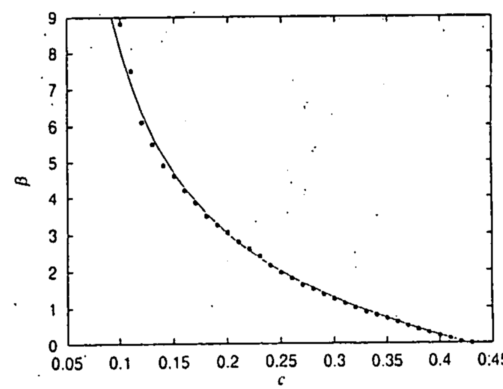

Fig. 14. Relation between β and c. Dots represent numerically estimated values and the curve shows the theoretical relation given by Eq. (28).

where β is an exponent of the power. By assuming that $Z(\rho)$ is expanded in the following way:

$$Z(\rho) = |\rho|^{\beta} \sum_{k=0}^{\infty} a_k \rho^k , \tag{26}$$

we obtain the consistency condition for the relation between coefficients of the βth-order moment of Eq. (9):

$$c^{\beta} \langle n^{\beta} \rangle = 1 . \tag{27}$$

By using Eq. (7) we have the following relation for β and c:

$$c^{\beta} \frac{1 - e^{-\gamma}}{e^{-\gamma}} \sum_{n=1}^{\infty} n^{\beta} e^{-\gamma n} = 1 . \tag{28}$$

For given c Eq. (28) gives a theoretical value of β. In order to check the validity of the above relation we compare the relation between β and c of Eq. (28) with results obtained by computer simulations in Fig. 14. We can confirm a nice fit. According to Montegna and Stanley [5] the distribution of averaged stock market price changes are well approximated by a symmetric power-law distribution with an exponent about $\beta = 1.4$. From Eq. (28) we immediately obtain the best parameter for describing the long tail of real economic data which is given by $c = 0.28$ from Fig. 14. At present we do not have any theory which can explain this value.

As known from the theory of stable distributions variance of this fluctuation diverges for $0 < \beta < 2$. From Fig. 14 we can confirm that $\beta \leqslant 2$ is satisfied for $c \geqslant c^*$. Namely, the divergence of second order moment of Δp observed in Fig. 12 is consistent with the theory of stable distributions.

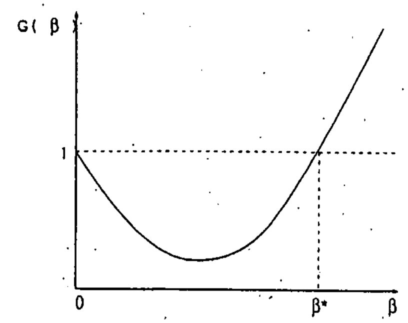

Fig. 15. A schematic illustration of the function $G(\beta)$, which has a minimum at $\beta \geqslant 0$. If $G'(0) < 0$ then there exists a solution of the equation, $G(\beta) = 1$.

Here, we discuss the existence of a solution β satisfying Eq. (27). Note that the function $G(\beta) \equiv c^{\beta} \langle n^{\beta} \rangle$ is continuous and convex with $G(0) = 1$ and $G''(\beta) > 0$ for $\beta > 0$. As known from a conceptual illustration, Fig. 15, the necessary condition to have positive β is given by $G'(0) < 0$.

$$\left| \frac{dG(\beta)}{d\beta} \right|_{\beta=0} = \langle \ln(cn) \rangle < 0 . \tag{29}$$

This condition can be recognized as the condition for the existence of stationary distribution. The necessary condition to have β in the range of Lévy stable distribution is therefore given by the following inequalities:

$$c^{*} < c < e^{-\langle \ln n \rangle} , \tag{30}$$

where

$$\langle \ln n \rangle = \frac{1 - e^{-\gamma}}{e^{-\gamma}} \sum_{n=1}^{\infty} (\ln n) e^{-\gamma n} . \tag{31}$$

By numerical estimation of Eq. (31) we have $e^{-\langle \ln n \rangle} \approx 0.42$. From Fig. 14 we can confirm the validity of this estimation by $\beta \approx 0$ near $c = 0.42$. For c larger than this value the fluctuation really diverges in each realization and we cannot define the steady-state distribution.

5. The cut-off in the distribution tails

In real phenomena showing power-law fluctuations there are always cut-offs, and there is a scale range in which a power law is a plausible approximation. In the cases of market price changes it is shown that the power-law distributions are accompanied with rapid decays for very large values [5]. We consider two reasons for such cut-off by using our model.

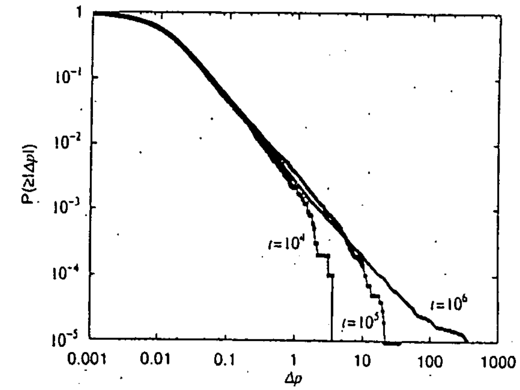

Fig. 16. Log–log plot of the cumulative distributions of Δp in the cases of finite observation times, $t = 10^4, 10^5$ and 10^6 with $c = 0.3$. Exponential cut-offs for large Δp are found for smaller t.

The first possibility is due to the finiteness of observation time. In order to demonstrate the effect of finite samples, we plot the distributions of Δp created by the stochastic model, Eq. (9) with different observation time intervals in Fig. 16. As clearly known from the figure for shorter observation time, the distribution tails decay more rapidly. A typical number of samples needed to confirm the power law for more than two decades is about of order of 10^5. This requirement may be too strong for real observation, namely, we cannot avoid this effect whenever we treat real data.

The other possibility is that there really exists a mechanism of the rapid decay. By introducing a nonlinearity or a kind of correlation between n_s and Δp_s we can realize such rapid decay [23]. We assume that n_s does not exceed a fixed value n_c for prices larger than a threshold price Δp_{th}, namely, $n_s < n_c$ for $|\Delta p| > \Delta p_{th}$. In Fig. 17 we show the cumulative distribution of Δp created by Eq. (18) with this modification. We can find the rapid decay around Δp_{th} as expected.

We can also modify our microscopic deterministic threshold model to have a rapid decay in the stationary power-law distributions. In the stochastic process the correlation between n and Δp can be introduced directly, however, this type of modification is not applicable in the microscopic deterministic threshold model because n is determined by the microscopic interactions. As a simplest modification to accommodate the correlation in the threshold model we assume that each dealer adds the last difference of market price to his bid price in geometric ratio until the next trade occurs when $|P(t) - P(t_{prev})| \geqslant \Delta p_{th}$ as follows:

$$\begin{aligned} B_i(t+1) &= B_i(t) + a_i(t) + c\{P(t) - P(t_{prev})\} , \\ B_i(t+2) &= B_i(t+1) + a_i(t+1) + c^2\{P(t) - P(t_{prev})\} , \\ B_i(t+3) &= B_i(t+2) + a_i(t+2) + c^3\{P(t) - P(t_{prev})\} , \\ &\vdots \end{aligned} \tag{32}$$

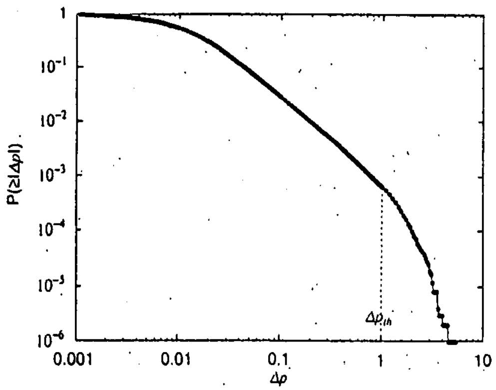

Fig. 17. Log–log plot of the cumulative distribution of Δp of the stochastic process (Eq. (9)) with an upper bound $n \leqslant n_c = 4$ for $|\Delta p| \geqslant \Delta p_{th} = 1$. $c = 0.3$.

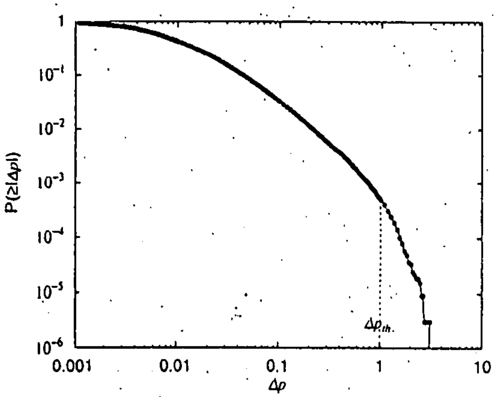

Fig. 18. Log–log plot of the cumulative distribution of Δp of the deterministic dealers model with damping memory (Eq. (32)). $\Delta p_{th} = 1, c = 0.3$.

This is a kind of memory damping effect and by this effect the price change involves a term $(1 - c^{n_s})/(1 - c)\Delta p_s$ instead of cn_s. As $(1 - c^{n_s})/(1 - c)$ converges to $1/(1 - c)$ in the limit of $n_s \to \infty$ for $c < 1$, this is effectively equivalent to suppose an upper limit for n in the case of large price changes. In Fig. 18 we can confirm a decay quite similar to the case of the modified macroscopic stochastic model, Fig. 17.

6. Conclusions and discussion

We introduced a very much simplified model of dealers in a market based on threshold dynamics, Eq. (3). We observed that the deterministic threshold model produces chaotic price changes spontaneously and the market-price changes are shown to be approximated by a simple stochastic equation, Eq. (9). From the macroscopic viewpoint the deterministic model can be effectively separated into two noise generators, an additive noise and a multiplicative noise. By analyzing these variables in detail we clarified relations between statistical quantities and the system's model parameters. With the assumption of independence between these additive noise and the multiplicative noise in Eq. (9) we have proved that an ideal price changes follow a power law in the steady state. The power law is ascribed to the Langevin-type equation with stochastic amplification. This mechanism is clearly different from the criticality of phase transition [24] in which a fine tuning of parameters are necessary to realize power-law distributions. Our systems have a resemblance to the self-organized criticality [25,26] in the meaning that power-law fluctuations appear spontaneously without any tuning.

We discussed the reason why the distributions of price changes in real stock market follow a symmetric power-law distribution accompanied with rapid decays for large variations. The first possibility is simply due to the finiteness of observation samples. The other possibility is the suppression of amplification, and examples of mechanism for both the macroscopic and the microscopic models have been proposed. A careful analysis of real data is required to judge which is more appropriate for real stock market.

In Fig. 1 the market-price evolution of threshold microscopic model have many impulses. We consider that the correlation between n_s and ϕ_s causes such impulses because there exists no impulses on market-price evolution of stochastic macroscopic model in which n_s and ϕ_s are independent. We need to modify our threshold model to accept correlation between n_s and ϕ_s. As we have seen our simplified models reproduce basic properties of stock market prices nicely, however, obviously the models dealing with only one brand are not complete. For example, we empirically know that a crash occurs simultaneously to almost all brands in real markets. We need to generalize our model to cope with interactions among brands.

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Stable Infinite Variance Fluctuations in Randomly Amplified Langevin Systems

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A general discrete stochastic process involving random amplification with additive external noise is analyzed theoretically and numerically. Necessary and sufficient conditions to realize steady power law fluctuations with divergent variance are clarified. The power law exponent is determined by a statistical property of amplification independent of the external noise. By introducing a nonlinear effect a stretched exponential decay appears in the power law. [S0031-9007(97)03737-X]

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Power law distributions have been found in diverse fields of science and the subjects of physicists' research are growing wider. Quantitative analyses are now in progress on a variety of topics showing power law distributions, for example, fish school sizes [1], frequency of jams in Internet traffic [2], and even stock market price changes [3]. In view of statistical physics the most important task is to elucidate general physical mechanisms underlying these power law behaviors.

A recently proposed concept of self-organized criticality is based on the idea that open systems showing power law distributions may have a mechanism of controlling inherent parameters automatically to be at the critical point of second order phase transition [4]. This idea has been confirmed theoretically for a sandpile model of avalanches that the critical point is stable in the renormalized macroscopic limit [5], and its applications are increasing rapidly.

Another general mechanism of producing power laws has been found in the study of stochastic processes involving multiplicative noises [6]. A typical equation of multiplicative process is given by a linear Langevin equation with a randomly changing coefficient. The effect of such a random coefficient has been intensively analyzed relating to the study of nonlinear dynamical systems because statistical properties of some nonlinear systems can be approximated by such stochastic equations [7,8]. It is intuitively obvious that multiplicative noises drastically enhance the additive random force in the Langevin equation and we have much larger fluctuations than in the case of constant coefficient. Numerical study and theoretical approaches strongly indicate the existence of a statistically steady state in which temporal fluctuations follow a power law distribution for a wide range of parameters in the random coefficient.

Theoretical analysis of the Langevin equation with a random coefficient is generally very difficult, because the master equation cannot be reduced to a solvable Fokker-Plank equation due to large fluctuations except in very special cases [9]. By this approach a sophisticated approximate theory has clarified that the Langevin equation with a random coefficient follows a steady distribution whose tails decay either following a stretched exponential form or a power law [10].

There is a powerful theoretical method for distributions with large fluctuations, the characteristic functions. The characteristic function is a Fourier transform of the probability density and the power law tails for an infinite variance distribution can be represented by a singularity at the origin of the corresponding characteristic function. The mathematical theory of stable distributions, which have power tails, is based on the characteristic functions [11], and some physical systems showing power law distributions have been solved rigorously by using characteristic function techniques [12].

In this paper we focus on temporal fluctuations having infinite variances. We introduce a discrete time version of the Langevin equation with a random coefficient and solve the steady state solution by introducing the characteristic function. We show rigorously that the tails of steady state probability density follow a power law in a very wide range of parameters. The necessary and sufficient conditions to realize the power laws with divergent variance is clarified; also the uniqueness and stability of the power law solution is proved theoretically. The exponent of the power law is not universal but changes continuously depending on the statistics of the coefficient. An exact formula is found for the exponent which clearly shows that the exponent is independent of the statistics of additive random force although the random force is necessary to realize the steady state. We confirm these results also by numerical simulations. In the final part of the paper we discuss briefly a possible direct application to the distribution of stock

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market price changes and rapid decays of distribution tails due to a nonlinear effect.

The model equation is given by the following simple discrete version of the linear Langevin equation:

$$x(t+1) = b(t)x(t) + f(t), \tag{1}$$

where $f(t)$ represents a random additive noise as usual, and $b(t)$ is a non-negative stochastic coefficient which means dissipation for $b(t) < 1$ and magnification for $b(t) > 1$. The case of magnification never occurs in a stable thermal equilibrium because it corresponds to "negative viscosity" in the continuum Langevin equation. However, magnification of fluctuations occurs in unstable systems in general; therefore, we believe Eq. (1) is a very basic starting point for general phenomena. In the following discussion, for simplicity, we assume that $b(t)$ and $f(t)$ are independent white noises having stationary statistics and $f(t)$ is symmetric.

Taking the average over the square of Eq. (1) we have the following equation for the second order moment:

$$\langle x^2(t+1)\rangle = \langle b^2\rangle\langle x^2(t)\rangle + \langle f^2\rangle, \tag{2}$$

where $\langle\cdots\rangle$ denotes an average over realizations.

As $\langle b^2\rangle$ and $\langle f^2\rangle$ are constants we can readily solve Eq. (2): For $\langle b^2\rangle < 1$ there is a stationary solution,

$$\langle x^2\rangle = \frac{\langle f^2\rangle}{1-\langle b^2\rangle}. \tag{3}$$

In the case of thermal equilibrium the principle of equipartition of energy requires that $\langle x^2\rangle$ is proportional to the temperature, so that $\langle f^2\rangle$ and $\langle b^2\rangle$ cannot be independent as known by the name of fluctuation-dissipation theorem [13]. For $\langle b^2\rangle > 1$ there is no stationary solution for $\langle x(t)^2\rangle$ and it diverges as $t \to \infty$. All higher moments diverge in the same way and it is common sense that such divergence means that the system is not statistically stationary. However, this common sense turns out to be wrong, as we prove in the following discussion. We have statistically steady fluctuation with infinite variance in the limit of $t \to \infty$.

Let the distribution functions of $b(t)$ and $f(t)$ be $W(b)$ and $U(f)$, respectively, which are assumed to be independent of time. The statistics of $x(t)$ is estimated theoretically by introducing the characteristic function, $Z(\rho,t)$, which is the Fourier transform of its probability density, $p(x,t)$:

$$Z(\rho,t) = \langle e^{i\rho x(t)}\rangle = \int_{-\infty}^{\infty} e^{i\rho x} p(x,t)\,dx. \tag{4}$$

Fourier transform of Eq. (1) gives the following basic equation for the characteristic function:

$$Z(\rho,t+1) = \langle e^{i\rho b(t)x(t)}\rangle\langle e^{i\rho f(t)}\rangle = \int_0^{\infty} W(b)Z(b\rho,t)\,db\,\Phi(\rho), \tag{5}$$

where $\Phi(\rho)$ is the characteristic function for the additive noise, $f(t)$. By assuming Taylor expansion around $\rho = 0$ Eq. (5) derives a set of equations for moments including Eq. (2) for the lowest order. When the variance diverges $Z(\rho,t)$ have singularity at $\rho = 0$ in the limit of $t \to \infty$ and Taylor expansion cannot be applied for the steady solution. In such a case the following fractional power term can be assumed for the lowest order term because the characteristic function is generally a continuous function [11],

$$Z(\rho,\infty) = 1 - \text{const} \times |\rho|^{\beta} + \ldots, \qquad 0 < \beta < 2, \tag{6}$$

which is equivalent to the assumption of power law tails in the probability distribution:

$$P(\geq|x|) \propto x^{-\beta}, \tag{7}$$

where $P(\geq|x|)$ represents the cumulative distribution defined as

$$P(\geq|x|) = \int_{-\infty}^{-|x|} p(x')\,dx' + \int_{|x|}^{\infty} p(x')\,dx'. \tag{8}$$

Introducing the steady solution's functional form, Eq. (6), into Eq. (5), we have the following consistency condition for the lowest order of ρ in the case that the additive noise's variance is finite and $\Phi(\rho)$ is expanded in integer powers of ρ.

$$\langle b^{\beta}\rangle = 1. \tag{9}$$

For a given distribution of $W(b)$ Eq. (9) can be viewed as the equation determining the value of the singularity exponent, β.

Noting that the function $G(\beta) \equiv \langle b^{\beta}\rangle$ satisfies $G(0) = 1$ and $G''(\beta) > 0$ for $\beta > 0$, we have the following necessary conditions in order to have β in the range of $(0, 2)$:

$$\lim_{\beta\to+0} G'(\beta) = \langle \ln b\rangle < 0, \tag{10}$$

$$G(2) = \langle b^2\rangle > 1. \tag{11}$$

The latter condition (11) is obviously the condition for the divergence of variance, $\langle x^2\rangle = \infty$. The former condition (10) corresponds to the requirement of stationarity, namely, if this inequality does not hold the magnification rate is so strong that we do not have a statistically steady state.

We can prove the uniqueness and stability of the steady solution (6) in the following way. Assuming the existence of a steady solution of Eq. (5) the deviation from the steady solution, $\tilde{Z}(\rho,t) \equiv Z(\rho,t) - Z(\rho)$, satisfies the same equation with a different boundary condition, $\tilde{Z}(0,t) = 0$. By taking absolute values of the equation we have an inequality:

$$|\tilde{Z}(\rho,t+1)| \leq \max\{|\tilde{Z}(\rho,t)|\}|\Phi(\rho)|, \tag{12}$$

where $\max\{\cdots\}$ shows the maximum value. Therefore, in the case $|\Phi(\rho)| < 1$ for $\rho \neq 0$, which is satisfied whenever the external noise is continuously distributed,

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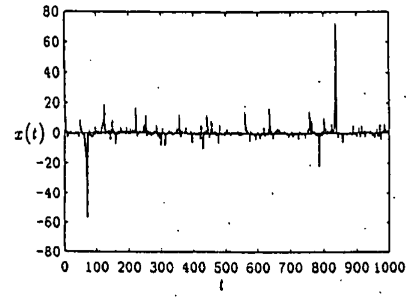

FIG. 1. An example of temporal fluctuations for $c = 0.3$.

the distribution of x converges quickly to the steady solution, (6), even starting from any initial distribution of $\{x(0)\}$. Namely, the conditions (10) and (11) are necessary and sufficient conditions for the power law with infinite variance.

Numerical simulation of Eq. (1) can be done easily. In order to specify the statistics we set the following distributions for b and f, respectively:

$$W(b) = \frac{1}{c} \sum_{k=0}^{\infty} (1 - e^{-\gamma}) e^{-\gamma k} \delta\left(\frac{b}{c} - k\right), \tag{13}$$

$$U(f) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-f^2/2\sigma^2}. \tag{14}$$

Here, the variable b takes a discrete value in $\{0, c, 2c, 3c, \ldots\}$ following the Poisson distribution, and f takes a continuous value following the symmetric Gaussian. The second order moment of b is given as $\langle b^2 \rangle = c^2 e^{-\gamma}(1 + e^{-\gamma})/(1 - e^{-\gamma})^2$, and the distribution $W(b)$ is controlled by a non-negative parameter c. In our numerical simulations the maximum time steps are typically 5×10^7 and we observe the distribution of $\{x\}$ for time steps after 1000. Figure 1 shows a typical example of temporal fluctuations for $\langle b^2 \rangle > 1$ with $\gamma = 0.32$, and $\sigma = 0.86$, which we chose for convenience of numerical calculations. As shown in Fig. 2 we can find a clear power law tail in the cumulative distribution of $x(t)$, $P(\geq |x|)$.

By repeating numerical calculations several times for each parameter we have confirmed that the power law exponent is independent of initial conditions, seeds of random number generator, and the functional forms of $U(f)$, as expected.

For different values of c the power law exponents are estimated numerically as shown in Fig. 3. In the case that distribution of b is given by Eq. (13) we can derive an analytic relation between β and c from Eq. (9) [14]:

$$c^{\beta}(1 - e^{-\gamma})\Gamma(\beta + 1)/\gamma^{\beta+1} = 1, \tag{15}$$

where $\Gamma(\beta)$ is the gamma function. We can confirm from Fig. 3 that Eq. (15) fits the numerical estimation quite nicely.

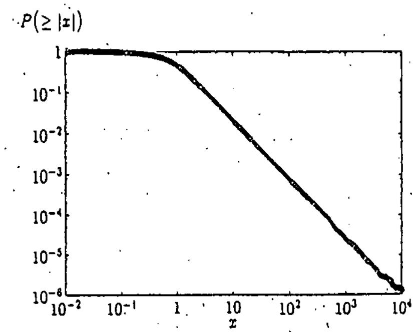

FIG. 2. Log-log plot of the cumulative distribution of x for $c = 0.3$.

It should be remarked that the theoretical estimate of Eq. (15) shows nice fits even out of the range of applicability, $\beta > 2$. The reason for this lucky coincidence is not clear but it is easy to tell that power law distribution tails are a generic property of Eq. (1). Actually, if the probability measure of $b(t) > 1$ is nonzero there exists a real number n_c and $\langle x(t)^n \rangle$ is divergent for $n \geq n_c$. This singularity implies that the distribution of $x(t)$ in the steady state has a power tail of Eq. (7) with $\beta = n_c$ [10]. On the contrary in the case of no magnification we have an analytic solution for $Z(\rho)$ and the distribution tails decay faster than any power.

Since the Langevin equation or its discrete version is one of the most basic stochastic equations not only in physics, potential applicability of our result is expected to be very wide. A direct application can be found in the cross-disciplinary field between statistical physics and economics. Stanley and his co-workers recently discovered nontrivial scaling relations in economic activities such as stock market prices [15]. It is pointed out that the distribution of averaged stock market price changes are well

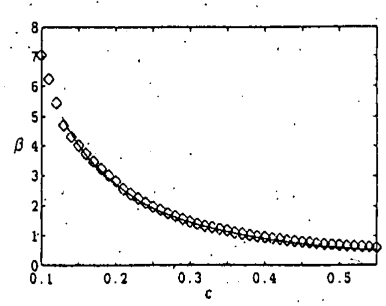

FIG. 3. The power law exponent β vs the amplification parameter c. Squares represent numerically estimated values and the curve gives the theoretical relation, Eq. (15).

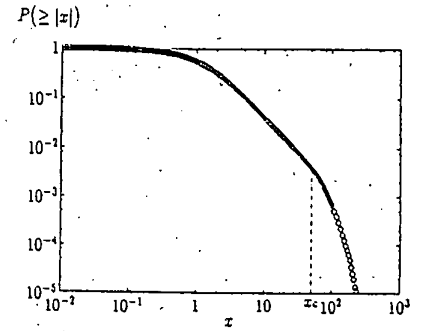

FIG. 4. Log-log plot of the cumulative distribution with the rapid decay. The threshold value is $x_c = 50.0$ with $c = 0.3$.

approximated by a symmetric power law distribution with an exponent about $\beta = 1.4$. The present authors (H. T. and A.-H. S.) have developed mathematical models of stock market prices and showed that price changes of a market can be approximated by Eq. (1) with the distribution of $b(t)$ given by Eq. (13) [14]. Intuitively a stock price change in a unit time is either magnified or damped randomly with an additive external noise. Our general result summarized in Fig. 3 immediately indicates that the best parameter for describing the long tail of real economic data can easily be estimated.

In real systems there is no rigorous power law distributions, but power tails are normally accompanied with rapid decays for very large values. Our model equation, Eq. (1), can easily be modified to manage this deviation from the power law. In Eq. (1) we assume that $b(t)$ is independent of $x(t)$; however, a system size limitation, for example, may introduce a correlation between $b(t)$ and $x(t)$. As a simplest modification we assume that $b(t) < 1$ for $|x(t)| > x_c$, where x_c is a given threshold value. In Fig. 4 a steady state distribution with this modification is shown. We can find a power law distribution with a rapid decay around x_c which can be approximated by a stretched exponential form.

Summarizing the results we have clarified necessary and sufficient conditions for a quantity described by Eq. (1) to follow a power law distribution with divergent moment. This is a new general route to power law fluctuations and it is now obvious that the divergence of variance is the most essential key ingredient for power law distributions.

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PHYSICA A

Critical fluctuations of demand and supply

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Abstract

In order to describe price changes in open markets we introduce a virtual balanced price which is determined by the distribution of dealers' expectation at a time. The dealers do not know directly the virtual balanced price but they can only guess it from the time series of market prices. By this assumption we derive a set of stochastic time evolution equations composed of the market price and the virtual balanced price as an extension of Langevin type equations.

1. Introduction

Price changes in open markets are very attractive research subject in view of statistical physics. First, each dealer's basic motion is rather simple, he buys when the price is expected to rise and he sells in the opposite situation. There is a universal aim of dealers, that is, to maximize his gain. Second, the market price fluctuates almost stochastically in time unit of seconds and all data are stored digitally in some databases which are directly analyzable by computers, namely no experimental set up is needed for the observation of data. A goal of statistical physics approach of price changes should be to derive and to understand the real market price fluctuations quantitatively from the microscopic dealer dynamics.

Statistical physics approach to price fluctuations appeared only recently. Mantegna reported that the distribution of market price change is approximated by a power law in 1991 [1]. One of the authors and coworkers introduced a model of market consisted of dealers having deterministic nonlinear interaction and showed that the price fluctuation can be caused by the effect of deterministic chaos in 1992 [2]. The power spectrum of the obtained fluctuation is characterized by an inverse square law, which is consistent

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with empirical data. It was also shown theoretically that the price change can be approximated by a Langevin type stochastic equation with random coefficient [3].

A new stage started with the paper of Mantegna and Stanley in 1995 in which they clearly demonstrated the existence of scaling relations in price changes and showed that the distribution of price changes can be approximated by a power law with cutoffs [4]. The paper attracted a number of physisists' interest in the price change problem especially on the power law behaviors. Levy and Solomon [5], Sornette and Cont [6], and the authors with Sato [7,8] investigated this problem and found that the essential mechanism of producing the large fluctuations following power law is the multiplicative stochasticity or the effect of random coefficient in Langevin equation.

In this paper we first discuss why prices are not stable in general from the view point of phase transition between the two phases of excess demand and excess supply [9]. Then we consider the balance of demand and supply in general open market and derive a set of stochastic price equations (B consisted of a virtual balanced price and market prices. The set of equations converges to the Langevin equation with random coefficients in a domain of parameters, but there exist cases that cannot be described by Langevin equation. Such basic properties of the price equation is studied in the last part of the paper.

2. Why prices fluctuate?

Everybody knows that prices are always fluctuating. But, why? There has been two physical approaches to answer this basic question.

An approach from microscopic viewpoint is to consider the properties of the elements, that is, the dealer's dynamics. Each dealer in a market has his expectation price in mind. He changes the price gradually by either analyzing the market price history or urged by some needs. A trade between two dealers occurs suddenly when their prices in minds agree. Namely, the interaction between dealers is more like a discontinuous step function than any continuous function, so it is very nonlinear. Also, the trade is irreversible in the sense that the reversed transaction never occurs at the same time. As the elements of market are highly nonlinear and irreversible, numerically simulated time evolution of market price almost always produces chaotic fluctuation [2]. In this sense the microscopic origin of price changes is in the nonlinear chaotic dynamics just like chaotic behaviors observed in many nonlinear physical systems.

Another approach is based on macroscopic view of demand and supply [9]. Usually demand and supply are regarded as function of price only in economics, but it is obvious that we cannot neglect the time dependence of these quantities. The quantity of supply in a unit time can be regarded as constant if we consider a commodity produced by factory for example, however, the quantity of demand is unavoidably stochastic because it is typically given by the number of consumers visiting a store in a unit time which is essentially unpredictable in liberalism economy. To take into account the effect of temporal fluctuation we introduce time-dependent demand and

supply, $\langle d(p,t)\rangle$ and $s(p,t)$, which represent newly emerged demand and supply in a time interval $(t-\Delta t, t)$. We introduce a cumulative demand minus supply at time t,

$$I(t)=\sum_{t'}^{t}\{d(p(t'),t')-s(p(t'),t')\} \quad (1)$$

The behaviors of $I(t)$ for a fixed price, p, can be categorized into the following three cases.

(1) Excess demand phase, $\langle d(p,t)\rangle > \langle s(p,t)\rangle$,
(2) Excess supply phase, $\langle s(p,t)\rangle > \langle d(p,t)\rangle$,
(3) The balanced point, $\langle d(p,t)\rangle = \langle s(p,t)\rangle$.

In the excess demand phase $I(t)$ fluctuates with positive trend, clearly this phase is not stable in the long time. The price p will be increased or the supply function will be increased in the future. Just the opposite motions are expected in the excess supply phase. At the balanced point $I(t)$ follows a Brownian motion if the intrinsic fluctuations of $d(p,t)$ or $s(p,t)$ is uncorrelated. As known immediately from Brownian motion's basic property the function $I(t)$ is not only almost always nonzero but it can take any large absolute value or it can keep the same sign for any long time period, namely, for example, there is a possibility that an excess demand state continues for a very long time. As no one knows whether the averaged quantities are balanced or not, such continuation of excess demand state may cause a price change to higher value, but such modification of price breaks the real balanced point so the new price should be modified again soon. As known from this discussion the variance of $I(t)$ at the balanced point is diverging and the balanced point is not stable in practical sense.

It should be noted that almost all goods displayed in shops are not at the balanced point but they are in the excess supply phase in rigorous sense. Here we define the rigorous balance by $I(t)=0$ or by the following meaning; as soon as goods are supplied they are sold immediately but there is no line of waiting consumers which is an indication of the excess supply state. Therefore, by this meaning all goods in ordinary shops or department stores are regarded as being in excess supply phase, actually many unsold goods are disposed after a while in most of these stores. It is nearly impossible to find goods in stores for which the $I(t)$ value is always close to zero.

The only exception that satisfy the balance condition, $I(t)=0$, rather nicely is the goods dealt in open market. In an open market a dealer who wants to sell gradually decreases his price in mind until he can find a person who wants to buy with the same price, and the reverse process occurs in the opposite situation. Therefore, the demand and supply are canceled immediately by changing the price and the expected value of $I(t)$ should be always close to 0 in an ideal market. The important point is that prices in open market changes very rapidly and randomly reflecting the intrinsic fluctuations in demand or supply.

As known from these discussions the physical mechanism of price fluctuation in a market is now clear. Microscopically the elementary process of trade is highly nonlinear and irreversible and such effects cause unpredictable motions among the dealers in the market. Macroscopically the balance of demand and supply which are given by the

numbers of buyers and sellers, can be realized at the critical point and the market price surrenders to the critical fluctuations. In the next section we derive a set of basic equations of price fluctuation following these ideas.

3. Derivation of the price equation

We assume an open market in which a huge number of dealers are trading such as the foreign exchange market. The basic assumptions in this section are the followings:

- Each dealer has buying and selling prices and he changes these prices in each time step.
- The dealers can only guess the future price from the past price data.
- The market price goes up when demand (number of buyers) is larger than supply (number of sellers) and it goes down in the opposite case.

In order to derive the basic equations from these assumptions we first generalize the quantity, the cumulative demand minus supply, $I(t)$, as a function of price and time, $I(p,t)$. This quantity shows the amount of demand minus supply at price p at time t and it is to be called as cumulative demand in short. Such function can be defined for each dealer, for example, when the buying and selling prices of the jth dealer at time t is given by $p_b(j,t)$ and $p_s(j,t)$, his contribution to the cumulative demand, $I(j,p,t)$, is described by the following function;

$$I(j,p,t)=\Theta(p_b(j,t)-p)-\Theta(p-p_s(j,t)), \quad (2)$$

where $\Theta(x)$ is the step function which is 0 for $x<0$ and is 1 for $x\geqslant 0$. For the whole market the cumulative demand, $I(p,t)$, is given by the summation of $I(j,p,t)$ over all j. It is a natural assumption that this function becomes a smooth function of p in the large market limit. In the stationary case the cumulative demand can be related to the ordinary demand and supply relations in classical economics theory. In economics the demand and supply prices, P_d and P_s, are characterized by the functions, $P_d=D(n_d)$ and $P_s=S(n_s)$, where n_d and n_s denote the quantity of demand and supply. We consider inverse functions, $D^{-1}(p)$ and $S^{-1}(p)$, then $I(p,t)$ is simply given by the difference of these functions.

$$I(p,t)=D^{-1}(p)-S^{-1}(p). \quad (3)$$

The true balanced point is given by the price $p^*(t)$ that satisfies $I(p^*(t),t)=0$. We call this balanced price as the virtual balanced price which is different from the real market price, $p(t)$, as the real market price is determined in each trade between two dealers who do not know the virtual balanced price. But the direction of price change should be governed by the cumulative demand for the whole market. Here, we simply assume that the price change is proportional to the cumulative demand, namely,

$$p(t+\Delta t)-p(t)\propto I(p(t),t). \quad (4)$$

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In the situation that $p(t)$ is close to $p^*(t)$ the cumulative demand can be expanded around the virtual balanced price and the market price change is approximated by the following linear equation.

$$p(t+\Delta t) = p(t) + A(t)(p^*(t) - p(t)). \qquad (5)$$

Here, $A(t)$ is given by the inverse of the slope of $I(p,t)$ at $p = p^*(t)$. This quantity is proportional to the price elasticity that is defined by the change of demand per unit change of price [9] which should always be positive. For larger $A(t)$ the market price changes larger.

Each dealer changes his prices in mind, $p_b(j,t)$ and $p_s(j,t)$; by observing the price history $\{p(t), p(t-\Delta t), \dots\}$ and the virtual balanced price in the next time step is determined from such information. Therefore, formally we have the next expression,

$$p^*(t+\Delta t) = p^*(t) + f(t) + F(p(t), p(t-\Delta t), \dots), \qquad (6)$$

where $f(t)$ represents a random variable having zero mean showing the statistical fluctuation of dealers' expectation. Determination of the functional form of the function $F(p(t), p(t-\Delta t), \dots)$ is a delicate problem because each dealer should have complicated strategy in his expectation of future prices, however, it is likely that the latest market price change, $p(t) - p(t-\Delta t)$, may give the largest contribution. Neglecting all other terms we propose the following equation as the first order approximation,

$$p^*(t+\Delta t) = p^*(t) + f(t) + B(t)(p(t) - p(t-\Delta t)). \qquad (7)$$

Here, $B(t)$ shows dealers' averaged response to the market price change which is expected to be positive for very small Δt. Combining Eqs. (5) and (7) we have a set of closed equation for $\{p(t)\}$ and $\{p^*(t)\}$ for given $\{f(t)\}$, $\{A(t)\}$ and $\{B(t)\}$.

4. Basic properties of the price equation

As the price equation includes stochastic term $f(t)$ and given functions $A(t)$ and $B(t)$, the behaviors of $p(t)$ and $p^*(t)$ are very complicated in general cases. An interesting quantity in this model is the discrepancy of the market price and the virtual balanced price, $\delta p(t) = p(t) - p^*(t)$. By subtracting Eq. (7) from Eq. (5) we have the following closed equation for $\delta p(t)$.

$$\delta p(t+\Delta t) \doteq \{1 - A(t)\}\delta p(t) + B(t)A(t-1)\delta p(t-\Delta t) - f(t). \qquad (8)$$

The basic properties of $\delta p(t)$ can be considered theoretically for the case of $f(t) = 0$. It is easily shown that $\delta p(t)$ goes to 0 when both of $A(t)$ and $B(t)$ are positive and less than 1. In such case we can expect that the market price and the virtual balanced price are always close even with nonzero $f(t)$. By substituting the relation that $p(t)$ and $p^*(t)$ are nearly equal into Eq. (7) we have the Langevin type equation with random coefficient $B(t)$ which is identical to the known basic equation of price changes [7,8].

However, for $B(t)$ or $A(t)$ larger than 1 there appear discrepancy with the known Langevin type equation.

It is easily confirmed that in the case of $B(t) > 1$ the price discrepancy $\delta p(t)$ increases exponentially with time, and in the case of $A(t) > 1$ we have oscillations in $\delta p(t)$. We guess that the time dependence of $A(t)$ and $B(t)$ are slower than that of $f(t)$ and we expect that such exponential growth and oscillation behaviors can be observed in real price changes.

There are several important points for application of this formulation to real data analysis. One is the value of Δt. Obviously Δt governs the time scale of the fluctuation of prices so we think that its value should be about the same as the value of autocorrelation time of the price fluctuations. If Δt is much smaller than the autocorrelation time, then we should include other terms in eq. (7) such as $p(t-\Delta t) - p(t-2\Delta t)$, or $p(t-2\Delta t) - p(t-3\Delta t)$.

Another point is the effect of higher orders in the cumulative demand, $I(p(t), t)$, in Eq. (4). This effect is necessary for the stabilization of price change when the price change grows exponentially or the oscillatory behavior becomes dominant. Practically this effect can be taken into account by replacing the linear term in Eq. (5) by an analytic function like the hyperbolic tangent.

The most important point of this formulation may be the way of determining the functions, $A(t)$ and $B(t)$. We should first develop the method of estimating these functions from real data and study their behaviors especially the correlation with the market prices. For example, it is likely that if the market price goes up so much by the exponential growth due to $B(t) > 1$, the value of $B(t)$ should decrease after a while. In order to establish such correlation we need to analyze a huge amount of real data.

Summarizing the results we have derived a new price equation theoretically which is a generalization of the known equation. We are now developing a method of estimating the parameters from real data and checking the validity using tick data of foreign exchange rate. The results will be reported somewhere in the near future.

5. Unlinked bibliography

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PATENT
450100-02608

IN THE UNITED STATES PATENT AND TRADEMARK OFFICE

Applicant(s)	:	Hideki TAKAYASU, et al.
Serial No.	:	09/611,896
For	:	PRICE FLUCTUATION PREDICTING DEVICE AND PREDICTING METHOD, PRICE FLUCTUATION WARNING DEVICE AND METHOD, AND PROGRAM PROVIDING MEDIUM
Filed	:	July 7, 2000
Art Unit	:	3624
Examiner	:	Richard C. Weisberger

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Gordon M. Kessler, Reg. No.38,511

Name of Applicant, Assignee or Registered Representative

Gordon Kessler

Signature

March 4, 2003

Date of Signature

AMENDMENT

Assistant Commissioner for Patents
Washington, D.C. 20231

Sir:

In response to the outstanding final Office Action dated December 11, 2002, please amend this application as follows.

IN THE CLAIMS:

Please add new claims 25-30 as follows:

--25. (New) A fluctuation predicting device for time-sequence data having ever-changing manner, comprising means of:

(a) holding/preserving theoretical models of correlation function of fluctuations for a plurality of state of the time-sequence data respectively;

(b) acquiring sampling data by sampling local portion of real time-sequence data;

(c) generating real correlation function based on the sampling data;

(d) selecting one of the theoretical models of the step (a) best matching to the real correlation function generated in the step (c), and judging one of the states regarding the real time-sequence data.

26. (New) A fluctuation predicting method for time-sequence data having ever-changing manner, comprising steps of:

(a) holding/preserving theoretical models of correlation function of fluctuations for a plurality of state of the time-sequence data respectively;

(b) acquiring sampling data by sampling local portion of real time-sequence data;

(c) generating real correlation function based on the sampling data;

(d) selecting one of the theoretical models of the step (a) best matching to the real correlation function generated in the step (c), and judging one of the states regarding the real time-sequence data.

27. (New) A fluctuation predicting method of new claim 26, wherein the theoretical model of the correlation function is generated based on the following:

the time-sequence data comprises an equilibrium point;

the equilibrium point provided based on value is provided by multiplying a first parameter to recent change value of the time-sequence data; and

value of the time-sequence data after time Δt is provided based on value provided by multiplying a second parameter to difference between value of the time-sequence data in current time t and the equilibrium point.

28. (New) A fluctuation predicting method of new claim 27, wherein the theoretical model of the correlation function is generated based on the following:

a unique corresponding relation is established between a pair of the first and second parameters and the correlation function.

29. (New) A fluctuation predicting method of new claim 27, wherein:

the time-sequence data represents data market price of an open market;

the equilibrium point represents virtual equilibrium prices;

the first parameter represents a reciprocal number of market instability of coefficients;

the second parameter represents a price resilience coefficient.

30. (New) A fluctuation predicting program for time-sequence data having an ever-changing manner, getting information processing device work as means of:

(a) holding/preserving theoretical models of correlation function of fluctuations for a plurality of state of the time-sequence data respectively;

(b) acquiring sampling data by sampling local portion of real time-sequence data;

(c) generating real correlation function based on the sampling data;

(d) selecting one of the theoretical models of the step (a) best matching to the real correlation function generated in the step (c), and judging one of the states regarding the real time-sequence data.

REMARKS

At paragraphs 1-8, the examiner has requested information regarding various articles and the like that were used in development of applicants' invention, and particularly in developing equations 1 and 2 and expressions 3-6. Applicants wish to notify the examiner of the following articles.

1. Aki-Hiro Sato, and Hideki Takayasu, "Dynamic numerical models of stock market price: from microscopic determinism to macroscopic randomness" Physica A, 250(1998), 231-252.

2. Hideki Takayasu, Aki-Hiro Sato, and Misako Takayasu, "Stable infinite variance fluctuations in randomly amplified Langevin systems" Phys. Rev. Lett,, 79(1997), 9676-969.

3. H. Takayasu and M.. Takaysu, "Critical fluctuations of demand and supply", Physica A 269 (1999) 24-29.

Additionally, Applicants note that additional internally generated, unpublished documents were reviewed by the inventors during the process of invention of the pending application. Copies of these non-public documents have not been retained.

Applicants generally submit that the claimed invention differ from the features set forth in the references in that the coefficients of formulae in the present invention are estimated from real data, but the coefficients in the references are generally randomly selected. Furthermore, the present invention uses simultaneous equations with two unknowns (A and B), however, references 1-2 failed to enclose these two simultaneous equations with two unknowns. Further differences between the claimed invention and the articles submitted by the Applicants include the following.

In the first article (Article No. 1) the following equation is introduced.

$$X(t+1)=b(t)x(t)+f(t).$$

And, in it, it is thought that fluctuation which is near to that of real market price can be obtained by regarding B(t) and f(t) as random variables. The coefficient b(t) is related to B(t) of this invention, and the thesis of the 1999 Physica A (Article No. 3). Though price equations are extended into simultaneous equations with two unknown in the second article this second article would arrive at the same above equation and the B(t)would become the b(t) if special limit is taken. However, this the version of the above equation cannot describe a vibration phenomenon of price at all. So considering this equation, it is very incomplete in its predictive capability as compared to the present invention.

In the second article (Article No. 2), at equ.(9) of its body, following equation is derived from Dealer Model, a market model.

$$\Delta p_{s+1}=cn_s\Delta p_s=+\o_2$$

However, s denotes time, Δp denotes price, so this article's equation would become the price equation of the Article No. 1 with one variable, if the term cn_s (the produce of constant c and random variable) is replaced into B(t). This price equation is therefore still considered as an equation with one variable.

Further, in the present invention coefficients A and B are estimated from real data but in the prior art, including the fourth article, coefficient B is determined in random manner and a method using estimation from real data is not described.

Therefore, Applicants submit that the claimed invention differs from these prior arts that were relied upon by these articles that were relied upon in development of the invention.

At paragraph 9 of the outstanding Office Action the Examiner has rejected claims 1-24 under 35 U.S.C. § 112 as being indefinite for failing to particularly point out and distinctly

claim the subject matter which the applicant regards as the invention. The Examiner states that the scope that price resilience indicator and market instability indicator are indefinite, lacking art recognized meanings.

Applicants wish to point out that a price resilience indication and market instability indicator are explained in the specification, regarding equations 1 and 2, and the explanation thereof at pages 23 -26 of the application as filed. Applicants further submit that the terms best regarding a best match and relatively small number is understood by one of ordinary skill in the art and that normally a substantially larger number of samples of data would be required to determine a correlation. Applicants therefore respectfully request that the rejection of claims 1-24 under 35 U.S.C. § 112 be withdrawn.

Applicants also present new claims 25-30 for consideration.

CONCLUSION

Applicants have made a diligent effort to provide information as required by the examiner. Early and favorable reconsideration of this information and the claimed invention are respectfully requested.

Respectfully submitted,

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